Compositional Analysis for Linear Control Systems

Kerber, Florian; Schaft, Arjan van der

Published in:
HSSC 10: PROCEEDINGS OF THE 13TH ACM INTERNATIONAL CONFERENCE ON HYBRID SYSTEMS: COMPUTATION AND CONTROL

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Florian Kerber
University of Groningen
P.O. Box 407
Groningen, the Netherlands
f.j.kerber@rug.nl

Arjan van der Schaft
University of Groningen
P.O. Box 407
Groningen, the Netherlands
A.J.van.der.Schaft@rug.nl

ABSTRACT

The complexity of physical and engineering systems, both in terms of the governing physical phenomena and the number of subprocesses involved, is mirrored in ever more complex mathematical models. While the demand for precise models is indisputable, the analysis of such system models remains challenging. Adopting techniques from computer science makes available a framework for compositional analysis of interconnected control systems. Simulation relations relate process models with their specifications thus checking whether the derived model behaves as desired. Based on that, compositional and assume-guarantee reasoning rules decompose the actual verification task into several subtasks that can be checked with less computational effort. Thus, modularly composed system models can be treated with modular analysis techniques. In this paper, we want to give an overview of how these concepts can be applied to analyze linear continuous-time systems (LTI). Motivated by the underlying physics, we introduce a general type of interconnection that can also be interpreted as a feedback control configuration in the spirit of decentralized control. Additionally, parallel composition of LTI systems is discussed with special emphasis on decomposition strategies for a given specification. The proposed methodology could be extended further to classes of hybrid systems where compositional analysis techniques are of particular interest.

Categories and Subject Descriptors

G.0 [**Mathematics of Computing**]: General

General Terms

Theory

Keywords

Linear Systems, Assume-Guarantee Reasoning, Compositional Reasoning, Simulation Relations

1. INTRODUCTION

In formal verification, the curse of dimensionality forms an obstacle to efficiently check properties of programs involving concurrent processes. Interaction between these concurrent processes leads immediately to combinatorial explosion in terms of the size of the state space. As a result, straightforward approaches to formal verification such as simulating all possible executions of a program usually fail. More structured techniques are needed instead to deal with the inherent complexity. One important development was the introduction of simulation relations by Milner [10]. Expressing both the program to be verified and the property to be checked in the same language – in the area of verification mostly as labeled transition systems – and then relating them by constructing a simulation relation ensures that the given system behavior matches the desired specification. To reduce the complex verification task for the overall system, compositional analysis techniques can be employed. The main idea of compositional reasoning is to decompose proof obligations for the whole interconnected system into obligations for components which computationally are more efficiently solvable. Complementary to compositionality is the idea of assume-guarantee reasoning which can be used when properties of individual components can not be verified directly ([11]). The key principle is to restrict the behavior of a subsystem to a specific environment by interconnecting it with a subsystem representing parts of the specification. Splitting the global proof obligation into several steps for restricted components, it is possible to guarantee the original verification goal yet with reduced efforts. Both for labeled transition systems and hybrid systems, there have been applications of compositional and assume-guarantee reasoning in recent years, see e.g. [4] and [5]. Encouraged by these advances in the area of computer science, compositional analysis techniques could play an important role for the analysis of control systems as well. In fact, models of engineering systems have similar features as models of concurrent processes. Firstly, the number of state components is large in several applications, e.g. for chemical plants, mechatronic or embedded systems ([3]). Secondly, interaction between subprocesses is characteristic for various control problems such as decentralized control where a global control target is solved by the interplay of local controllers and plant subsystems. The goal of this work is to make compositional techniques applicable to analyze linear continuous-time systems. As a first step, simulation relations for dynamical systems have been introduced, see in particular [1], [12] and [14]. Besides using them to verify properties of implemented process models, simulation relations can also serve as a tool to abstract a given model with a lower dimensional one. This idea was brought forward in [13] as a means to reduce the complexity of interconnected system models. Abstractions provide a conservative approximation of the given system model so that properties can be checked reliably on a higher level. A two-sided version of simulations, bisimulation relations, has been studied extensively in [12] for both labeled transition systems and continuous-time control systems stressing the link between formal verification and control theory. Moreover, the idea of compositional reasoning has recently been investigated for feedback interconnections of linear ([7]) and hybrid systems ([6]). This paper generalizes and extends the proposed methodology by considering two different types of interconnection for linear continuous-time systems. Motivated by many physical applications, a feedback type of interconnection is studied first where the external variables are equated. A methodology for compositional and assume-guarantee reasoning is developed and illustrated with an example from circuit analysis. Second, parallel composition is introduced as an alternative but equally relevant interconnection for control systems. We focus on decomposition strategies for a given global specification, i.e. how to arrive at an interconnection of local specifications that can then be used for compositional and assume-guarantee reasoning. We conclude by giving an outlook as to possible further directions of research.

2. PRELIMINARIES

Consider the class of linear continuous-time systems

$$\begin{aligned}\Sigma_i: \quad \dot{x}_i &= A_i x_i + B_i u_i + G_i e_i + L_i d_i \\ y_i &= C_i x_i \\ z_i &= H_i x_i\end{aligned} \tag{1}$$

All variables belong to finite dimensional vector spaces, $x_i \in \mathcal{X}_i, u_i \in \mathcal{U}_i, e_i \in \mathcal{E}_i, d_i \in \mathcal{D}_i, y_i \in \mathcal{Y}_i, z_i \in \mathcal{Z}_i$. The temporal evolution of all system variables is characterized by functions of an appropriate function class, e. g. $\mathcal{C}^\infty$. The variables u_i and y_i are used for interconnections, e_i and z_i are control inputs and outputs and d_i represents an (internal) disturbance.

Remark 1. *Systems with disturbance inputs (called 'non-deterministic systems' in [13]) arise naturally from abstractions. An abstraction of a dynamical system – very similar to and inspired by abstractions for programs of concurrent processes in computer science – incorporates a generator of non-determinism that allows to preserve the properties of interest while reducing the complexity of a model. As proposed in [12] and [14], a non-deterministic system of the form* (1) *can therefore abstract another linear control system of higher state space dimension.*

Compositional analysis, the main focus of this research, depends heavily on the type of interconnection. Physical systems are usually interconnected by equating the interconnection variables, for example forces and positions in mechanical systems, currents and voltages in electrical circuits or pressure and volume in chemical reactors. In this paper, we study two particular cases for systems Σ_i: The first one is the standard feedback interconnection where one system represents the plant and the other teh controller. The other type of interconnection is parallel composition which will be detailed in Section 4.

Definition 1. *Two linear continuous-time systems Σ_1, Σ_2 of the form (1) are interconnected by equating the interconnection variables,*

$$u_2 = y_1 \ , \ u_1 = y_2 \ , \tag{2}$$

The dynamics of the interconnection $\Sigma_1 \| \Sigma_2$ are then given by

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}A_1 & B_1C_2\\ B_2C_1 & A_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \\ &\quad + \begin{bmatrix}G_1 & 0\\ 0 & G_2\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix} + \begin{bmatrix}L_1 & 0\\ 0 & L_2\end{bmatrix}\begin{bmatrix}d_1\\ d_2\end{bmatrix}\\ \begin{bmatrix}z_1\\ z_2\end{bmatrix} &= \begin{bmatrix}H_1 & 0\\ 0 & H_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\end{aligned} \tag{3}$$

Remark 2. *Equating shared variables is also common when modeling compositions of concurrent systems, see e. g. the tagged signal model framework as presented in [9].*

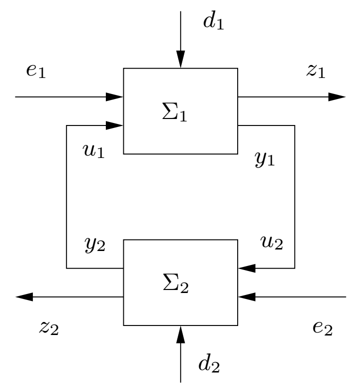

Figure 1: Interconnection $\Sigma_1 \| \Sigma_2$

Remark 3. (3) *it is a generalization of the feedback interconnections in [7]. More specifically, the interconnection variables u_1 represent a second channel of inputs which is in general independent of the external inputs e_1. The open feedback interconnection in [7] is a special case of (3) where $G_i = -B_i$ as illustrated in Figure 2 while closed feedback interconnection would reduce to $G_i = 0$. A typical application*

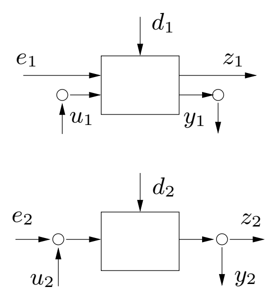

Figure 2: Generalization of feedback interconnections

for compositions of feedback control configurations are decentralized control problem, e. g. the control of robot networks or communication of distributed sensors.

In order to verify that a given system model fulfils its specification, i.e. behaves in a desired fashion, the concept of simulation relations proves valuable.

Definition 2. *A simulation relation S of Σ_1 by Σ_2 is a linear subspace $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ with the following property: For any $(x_{10}, x_{20}) \in S$, any joint control input function $e_1(\cdot) = e_2(\cdot) = e$, any joint interconnection input $u_1(\cdot) = u_2(\cdot) = u(\cdot)$ and any disturbance function $d_1(\cdot)$ there should exist a disturbance $d_2(\cdot)$ such that the resulting state trajectories $x_i(.), i = 1, 2$ with $x_i(0) = x_{i0}$, satisfy for all $t \geq 0$*

$$\begin{aligned} &(i) \quad (x_1(t), x_2(t)) \in S,\ \forall t \geq 0 \\ &(ii) \quad H_1 x_1(t) = H_2 x_2(t),\ \forall t \geq 0 \\ &(iii) \quad C_1 x_1(t) = C_2 x_2(t),\ \forall t \geq 0 \end{aligned} \tag{4}$$

Σ_1 is simulated by Σ_2, denoted by $\Sigma_1 \preccurlyeq \Sigma_2$, if there exists a simulation relation S fulfilling $\Pi_1 S = \mathcal{X}_1$ with $\Pi_1 : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{X}_1$ the canonical projection from $\mathcal{X}_1 \times \mathcal{X}_2$ to $\mathcal{X}_1$. In this case, S is called a full simulation relation.

Proposition 1. *A subspace $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a simulation relation of Σ_1 by Σ_2 if and only if for all $(x_1, x_2) \in S$, all $u \in \mathcal{U}$ and all $e \in \mathcal{E}$ the following holds:*

(i): for all $d_1 \in \mathcal{D}_1$ there should exist a $d_2 \in \mathcal{D}_2$ such that

$$\begin{bmatrix} A_1 x_1 + B_1 u + G_1 e + L_1 d_1 \\ A_2 x_2 + B_2 u + G_2 e + L_2 d_2 \end{bmatrix} \in S \tag{5}$$

(ii): $\quad H_1 x_1 = H_2 x_2$

(iii): $\quad C_1 x_1 = C_2 x_2$

Invariant subspaces as used in geometric control theory allow us to formulate an equivalent linear algebraic characterization of a simulation relation.

Theorem 1. *A linear subspace $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a simulation relation of Σ_1 by Σ_2 if and only if the following holds:*

(i): $\quad \operatorname{im}\begin{bmatrix} G_1 & B_1 \\ G_2 & B_2 \end{bmatrix} + \operatorname{im}\begin{bmatrix} L_1 \\ 0 \end{bmatrix} \subset S + \operatorname{im}\begin{bmatrix} 0 \\ L_2 \end{bmatrix}$

(ii): $\quad \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} S \subset S + \operatorname{im}\begin{bmatrix} 0 \\ L_2 \end{bmatrix}$

(iii): $\quad S \subset \ker\begin{bmatrix} H_1 & -H_2 \\ C_1 & -C_2 \end{bmatrix}$

The linear algebraic characterization of Theorem 1 facilitates an effective algorithm how to compute the maximal simulation relation of Σ_1 by Σ_2.

Theorem 2 (compare with Theorem 3.4 in [14]). *For two linear systems Σ_1 and Σ_2, define the following sequence of decreasing subspaces $S^i, i = 1, \ldots$:*

$$\begin{aligned} (i): \quad S^1 &= \ker\begin{bmatrix} H_1 & -H_2 \\ C_1 & -C_2 \end{bmatrix} \\ (ii): \quad S^i &= \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in S^{i-1} \;\middle|\; \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right. \\ &\quad \left. + \operatorname{im}\begin{bmatrix} L_1 \\ 0 \end{bmatrix} \in S^{i-1} + \operatorname{im}\begin{bmatrix} 0 \\ L_2 \end{bmatrix} \right\} \\ &\quad \forall i = 1, \ldots, k \end{aligned}$$

If for a certain i the subspace S^i is empty, then there does not exist any simulation relation of Σ_1 by Σ_2. Otherwise, there exists a finite k such that $S^k = S^{k-1} =: S^\star$. Then there exists a simulation relation of Σ_1 by Σ_2 if and only if

$$\operatorname{im}\begin{bmatrix} G_1 & B_1 \\ G_2 & B_2 \end{bmatrix} \subset S^\star . \tag{6}$$

Furthermore, the subspace $S^\star$ is the maximal simulation relation of Σ_1 by Σ_2.

An important property of simulation relations is *transitivity*. This will become evident when non circular assume guarantee reasoning is discussed. First, we extend the well known results that simulation relations for labeled transition systems are preorders and that the interconnection $\|$ is symmetric with respect to simulation to linear control systems.

Theorem 3. *Simulation relations $\preccurlyeq$ are preorders, i.e. they are reflexive and transitive.*

Proof. Consider linear systems $\Sigma_i, i \in \{1, 2, 3\}$ of the form (1). *Reflexivity*: The relation $S = \{(x_1, x_1) \mid x_1 \in \Sigma_1\}$ fulfils conditions (i) and (ii) of Definition 2 and therefore defines a simulation relation of Σ_1 by Σ_1.
Transitivity: Assume S_1 defines a simulation relation for $\Sigma_1 \preccurlyeq \Sigma_2$ and S_2 for $\Sigma_2 \preccurlyeq \Sigma_3$. Then $S_{12} = \{(x_1, x_3) \mid \exists x_2 : (x_1, x_2) \in S_1, (x_2, x_3) \in S_2\}$ defines a full simulation relation of Σ_1 by Σ_3. □

Proposition 2. *For any two given linear systems Σ_P and Σ_Q,*

$$\Sigma_P \| \Sigma_Q \preccurlyeq \Sigma_Q \| \Sigma_P \tag{7}$$

Proof. Construct the relation

$$S = \{((x_P, x_Q), (x_Q, x_P)) \mid x_P \in \mathcal{X}_P, x_Q \in \mathcal{X}_Q\}$$

Since there exists a similarity transform $T = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ between $\Sigma_P \| \Sigma_Q$ and $\Sigma_Q \| \Sigma_P$ it is immediately seen that the interconnection $\|$ is indeed commutative with respect to simulation. □

3. COMPOSITIONAL AND ASSUME-GUARANTEE REASONING FOR LINEAR SYSTEMS

Consider the verification task

$$\Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2} \tag{8}$$

The interconnection $\Sigma_{P_1} \| \Sigma_{P_2}$ represents a given system model such as a feedback control configuration whereas $\Sigma_{Q_1} \| \Sigma_{Q_2}$ specifies a desired property or system behavior.

3.1 Compositional reasoning

The main pillar for compositional analysis is the so called *compositionality* property.

Theorem 4. *For any given linear systems Σ_i, $i \in \{P_1, P_2, Q_1, Q_2\}$, the* compositionality *property*

$$\begin{gathered} \Sigma_{P_1} \preccurlyeq \Sigma_{Q_1}, \Sigma_{P_2} \preccurlyeq \Sigma_{Q_2} \\ \Longrightarrow \\ \Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2} \end{gathered}$$

holds.

PROOF. Let $S_i, i = 1,2$ denote the full simulation relations of Σ_{P_i} by Σ_{Q_i}. Construct the relation

$$S = \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid (x_{P_1}, x_{Q_1}) \in S_1, (x_{P_2}, x_{Q_2}) \in S_2\} \quad (9)$$

Then for every $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$, every joint input $e_{P_i} = e_{Q_i} = e_i, i = 1,2$ and every disturbance $\begin{bmatrix} d_{P_1} & d_{P_2} \end{bmatrix}^T$ there exist disturbances d_{Q_1}, d_{Q_2} such that

$$\begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}C_{P_2}x_{P_2} + G_{P_1}e_1 + L_{P_1}d_{P_1} \\ A_{Q_1}x_{Q_1} + B_{Q_1}C_{Q_2}x_{Q_2} + G_{Q_1}e_1 + L_{Q_1}d_{Q_1} \end{bmatrix} \in S_1$$

and

$$\begin{bmatrix} A_{P_2}x_{P_2} + B_{P_2}C_{P_1}x_{P_1} + G_{P_2}e_2 + L_{P_2}d_{P_2} \\ A_{Q_2}x_{Q_2} + B_{Q_2}C_{Q_1}x_{Q_1} + G_{Q_2}e_2 + L_{Q_2}d_{Q_2} \end{bmatrix} \in S_2$$

whilst $H_{P_i}x_{P_i} = H_{Q_i}x_{Q_i}$ since $C_{P_i}x_{P_i} = C_{Q_i}x_{Q_i}$ for all $(x_{P_i}, x_{Q_i}) \in S_i$.
Moreover, S as defined in (9) is in fact the product of the simulation relations S_1 and S_2 after exchanging the second with the third component, i.e. reordering the elements x_{Q_1} and x_{P_2}. Since $\Pi_{P_1}S_1 = \mathcal{X}_1$ and $\Pi_{P_2}S_2 = \mathcal{X}_2$, i.e. S_1 and S_2 are full, also $\Pi_{P_1P_2}S = \mathcal{X}_1 \times \mathcal{X}_2$ and therefore S is full. □

REMARK 4. *The converse does in general not hold. Take as a counterexample the following systems*

$$\begin{aligned} \Sigma_{P_1} : \dot{x}_{P_1} &= 2u_{P_1} + e_{P_1} \\ y_{P_1} &= z_{P_1} = x_{P_1} \\ \Sigma_{P_2} : \dot{x}_{P_2} &= u_{P_2} + e_{P_2} \\ y_{P_2} &= \frac{1}{2}x_{P_2} \\ z_{P_2} &= x_{P_2} \\ \Sigma_{Q_i} : \dot{x}_{Q_i} &= u_{Q_i} + e_{Q_i} \\ y_{Q_i} &= z_{Q_i} = x_{Q_i} \end{aligned}$$

Then there exists a simulation relation S of $\Sigma_{P_1}\|\Sigma_{P_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$,

$$S = \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid x_{P_1} = x_{Q_1}, x_{P_2} = x_{Q_2}\}$$

since the state space descriptions of $\Sigma_{P_1}\|\Sigma_{P_2}$ and $\Sigma_{Q_1}\|\Sigma_{Q_2}$ are identical. On the contrary, there do not exist any simulation relations of Σ_{P_1} by Σ_{Q_1} nor of Σ_{P_2} by Σ_{Q_2} since for the former,

$$\mathrm{im}\begin{bmatrix} B_{P_1} \\ B_{Q_1} \end{bmatrix} = \mathrm{im}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \not\subseteq \ker\begin{bmatrix} C_{P_1} & -C_{Q_1} \\ H_{P_1} & -H_{Q_1} \end{bmatrix} = \mathrm{im}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and for the latter,

$$\mathrm{im}\begin{bmatrix} B_{P_2} \\ B_{Q_2} \end{bmatrix} = \mathrm{im}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \not\subseteq \ker\begin{bmatrix} C_{P_2} & -C_{Q_2} \\ H_{P_2} & -H_{Q_2} \end{bmatrix} = \mathrm{im}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As a special case of compositionality, *invariance under composition* also holds:

$$\forall \Sigma_{Q_2} : \quad \Sigma_{P_1} \preccurlyeq \Sigma_{Q_1} \implies \Sigma_{P_1}\|\Sigma_{Q_2} \preccurlyeq \Sigma_{Q_1}\|\Sigma_{Q_2} \quad (10)$$

In fact, since the interconnection $\|$ is commutative, compositionality and invariance under composition are equivalent.

PROPOSITION 3. *For any given systems Σ_i, $i \in \{P_1, P_2, Q_1, Q_2\}$ and a commutative interconnection, compositionality and invariance under composition are equivalent.*

PROOF. $(\Longrightarrow)$: Composing $\Sigma_{P_1} \preccurlyeq \Sigma_{Q_1}$ with Σ_{P_2} yields $\Sigma_{P_1}\|\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|\Sigma_{P_2}$ while composing $\Sigma_{P_2} \preccurlyeq \Sigma_{Q_2}$ with Σ_{Q_1} results in $\Sigma_{P_2}\|\Sigma_{Q_1} \preccurlyeq \Sigma_{Q_2}\|\Sigma_{Q_1}$. Using both commutativity and transitivity, invariance under composition follows. $(\Longleftarrow)$: Due to simulation being reflexive, compositionality immediately follows from invariance under composition,

$$\Sigma_{P_1} \preccurlyeq \Sigma_{Q_2}, \Sigma_{P_2} \preccurlyeq \Sigma_{P_2} \Longrightarrow \Sigma_{P_1}\|\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|\Sigma_{P_2}$$

□

3.2 Assume-guarantee reasoning

Since compositionality is in general not necessary and sufficient, i.e., it is not always possible to conclude from $\Sigma_{P_1}\|\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|\Sigma_{Q_2}$ that also the components fulfil their respective specifications, $\Sigma_{P_i} \preccurlyeq \Sigma_{Q_i}$, assume-guarantee reasoning can provide an alternative decomposition strategy. Again, the global proof obligation (8) is split into tasks for subsystems, but these components are now restricted by their environment, i.e. they are interconnected with other components. The first example are two non circular assume-guarantee reasoning rules which are based on only one unrestricted assumption yielding a triangular structure.

THEOREM 5. *For any given linear systems Σ_i, $i \in \{P_1, P_2, Q_1, Q_2\}$, non circular assume-guarantee reasoning is sound, i.e.*

$$\begin{array}{lrcl} S_1: & \Sigma_{P_1} & \preccurlyeq & \Sigma_{Q_1} \\ S_2: & \Sigma_{Q_1} \| \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \\ \hline S: & \Sigma_{P_1} \| \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \end{array}$$

and the symmetric counterpart

$$\begin{array}{lrcl} S_1': & \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_2} \\ S_2': & \Sigma_{P_1} \| \Sigma_{Q_2} & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \\ \hline S: & \Sigma_{P_1} \| \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \end{array}$$

hold.

PROOF. Notice first that rules 1 and 2 are symmetrical in their triangular structure. The proof only requires the relation $\preccurlyeq$ to be transitive and the interconnection $\|$ to be invariant under composition. For rule 1, interconnecting both Σ_{P_1} and Σ_{Q_1} in S_1 with Σ_{P_2} yields

$$\Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_2} \| \Sigma_{Q_2}$$

Similarly, interconnecting S_1' with Σ_{P_1} and exploiting symmetry of the interconnection results in

$$\Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{P_1} \| \Sigma_{Q_2} \preccurlyeq \Sigma_{Q_2} \| \Sigma_{P_1} \preccurlyeq \Sigma_{Q_2} \| \Sigma_{Q_2}$$

□

EXAMPLE 1. *Consider the LC$-$ circuit in Figure 3 with two inductors L_1 and L_2, one inductor C, a voltage source as input u_{P_1} and the current over the capacitor as output y_{P_1}. The control in- and outputs are chosen to be the same as the interconnection variables, $u_{P_1} = e_{P_1}$ and $y_{P_1} = z_{P_1}$, while external disturbances are absent, $d_{P_1} \equiv 0$.. The math-*

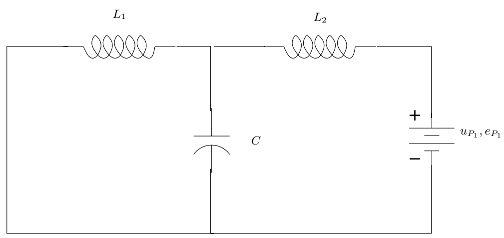

Figure 3: Σ_{P_1}: $LC-$ circuit.

ematical model Σ_{P_1} is given by

$$
\begin{aligned}
\Sigma_{P_1}: \begin{bmatrix} \dot{q}_C \\ \dot{\phi}_{L_1} \\ \dot{\phi}_{L_2} \end{bmatrix} &= \begin{bmatrix} 0 & \frac{1}{L_1} & \frac{1}{L_2} \\ -\frac{1}{C} & 0 & 0 \\ \frac{1}{C} & 0 & 0 \end{bmatrix} \begin{bmatrix} q_C \\ \phi_{L_1} \\ \phi_{L_2} \end{bmatrix} + \\
&+ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_{P_1} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e_{P_1} \\
y_{P_1} &= \begin{bmatrix} \frac{1}{C} & 0 & 0 \end{bmatrix} \mathbf{x}_{P_1} = z_{P_1}
\end{aligned}
$$

where $\mathbf{x}_{P_1} = \begin{bmatrix} q_C & \phi_{L_1} & \phi_{L_2} \end{bmatrix}^T$ denotes the state vector. In the remainder, all the parameter values are set to 1. To stabilize the electrical circuit (11) we apply a simple feedback controller Σ_{P_2},

$$
\begin{aligned}
\Sigma_{P_2}: \quad \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_{P_2} \\
y_{P_2} &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}
\end{aligned}
$$

Observe that we take $e_{P_2} = z_{P_2} = d_{P_2} \equiv 0$.
The verification goal is to relate the 5-dimensional interconnection of $\Sigma_{P_1} \| \Sigma_{P_2}$ to a less complex specification $\Sigma_{Q_1} \| \Sigma_{Q_2}$. The components of the specification are described by a non-deterministic $LC-$ circuit Σ_{Q_1} as in Figure 1 and an abstracted controller Σ_{Q_2}. The respective equations of motions

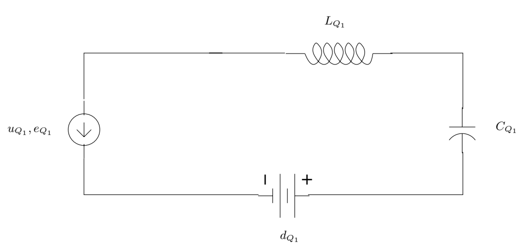

Figure 4: Σ_{Q_1}

are given by

$$
\begin{aligned}
\Sigma_{Q_1}: \begin{bmatrix} \dot{\phi}_{Q_1} \\ \dot{q}_{Q_1} \end{bmatrix} &= \begin{bmatrix} 0 & -\frac{1}{C_{Q_1}} \\ \frac{1}{L_{Q_1}} & 0 \end{bmatrix} \begin{bmatrix} \phi_{Q_1} \\ q_{Q_1} \end{bmatrix} + \\
&\begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{Q_1} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e_{Q_1} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d_{Q_1} \\
y_{Q_1} &= \begin{bmatrix} 0 & \frac{1}{C_{Q_1}} \end{bmatrix} \mathbf{x_{Q_1}} = z_{Q_1}
\end{aligned}
$$

where $\mathbf{x}_{Q_1} = \begin{bmatrix} \phi_{Q_1} & q_{Q_1} \end{bmatrix}^T$ and all parameter values are again set to 1. The controller Σ_{Q_2} is described by

$$
\begin{aligned}
\Sigma_{Q_2}: \dot{x}_{Q_2} &= -5x_{Q_2} + u_{Q_2} + d_{Q_2} \\
y_{Q_2} &= x_{Q_2}
\end{aligned}
$$

The first observation is that compositionality is not applicable since there does not exist any simulation relation of Σ_{P_1} by Σ_{Q_1}. The disturbance input d_{Q_1} represents a voltage source which cannot mimic the behavior of the inductor L_2. However, the controller systems Σ_{P_2} and Σ_{Q_2} can be related by means of a full simulation relation S_1',

$$S_1' = \{(z_1, z_2), x_{Q_2}) \mid z_1 = x_{Q_2}\}$$

Moreover, the interconnection $\Sigma_{P_1} \| \Sigma_{Q_2}$ can be simulated by $\Sigma_{Q_1} \| \Sigma_{Q_2}$,

$$
\begin{aligned}
S_2' = \{ & ((q_C, \phi_{L_1}, \phi_{L_2}, x_{Q_2}), (x_1, x_2, x'_{Q_2})) \mid x_{Q_2} = x'_{Q_2}, \\
& q_C = x_2, -1/5 q_C + 1/5 \phi_{L_1} + \phi_{L_2} + x_{Q_2} = x_1\}
\end{aligned}
$$

By Theorem 5, we can therefore conclude that there indeed exists a full simulation relation S of $\Sigma_{P_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$, given by

$$
\begin{aligned}
S = \{ & ((q_C, \phi_{L_1}, \phi_{L_2}, z_1, z_2), (x_1, x_2, x_{Q_2})) \mid z_2 = x_{Q_2}, \\
& q_C = z_2, q_C - \phi_{L_1} + \phi_{L_2} + x_1 = z_1\}
\end{aligned}
$$

This also shows that it is possible to abstract the behavior of the 5 dimensional controlled electrical circuit by a non-deterministic 3-dimensional electrical circuit.

The second assume-guarantee reasoning rule involves circular dependencies of assumptions of preceding steps and guarantees of successive steps in the proof. For the non-deterministic case, a proof for soundness of circular assume-guarantee reasoning is given in [8]. For ease of presentation, however, we will restrict ourselves in this paper to the deterministic case, i.e. to $d_i \equiv 0$. We first state the following auxiliary results to construct full simulation relations for interconnections of subsystems.

LEMMA 1. *Given full simulation relations S_1 and S_2 of $\Sigma_{P_1} \| \Sigma_{Q_2}$ and $\Sigma_{Q_1} \| \Sigma_{P_2}$ by $\Sigma_3 \| \Sigma_4$, respectively, then also*

$$S_1^{sym} := S_1 + \hat{S}_1, \quad S_2^{sym} := S_2 + \hat{S}_2 \tag{11}$$

define full simulation relations of $\Sigma_{P_1} \| \Sigma_{Q_2}$ and $\Sigma_{Q_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$, where

$$
\begin{aligned}
\hat{S}_1 &:= \{(x_{P_1}, \bar{x}_{Q_2}, x_{Q_1}, x_{Q_2}) \mid (x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in S_1\} \quad (12) \\
\hat{S}_2 &:= \{(\bar{x}_{Q_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid (x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \in S_2\}
\end{aligned}
$$

PROOF. We will only prove that S_1^{sym} is a simulation relation, the same reasoning can be applied to S_2^{sym}. Let S_1 be a simulation relation of $\Sigma_{P_1} \| \Sigma_{Q_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$ and consider an arbitrary element $(x_{P_1}, \tilde{x}_{Q_2}, x_{Q_1}, x_{Q_2}) \in S_1^{\text{sym}}$. Since S_1 is a simulation relation, it follows that

$$C_{Q_2} x_{Q_2} = C_{Q_2} \tilde{x}_{Q_2} \tag{13}$$

Then $(x_{P_1}, x_{Q_2}, x_{Q_1}, \tilde{x}_{Q_2}) \in S_1$. Hence for any input $\begin{bmatrix} e_1 & e_2 \end{bmatrix}^T$,

$$
\begin{bmatrix} \dot{x}_{P_1} \\ \dot{x}_{Q_2} \\ \dot{x}_{Q_1} \\ \dot{\tilde{x}}_{Q_2} \end{bmatrix} = \begin{bmatrix} A_{P_1} x_{P_1} + B_{P_1} C_{Q_2} x_{Q_2} + G_{P_1} e_1 \\ A_{Q_2} x_{Q_2} + B_{Q_2} C_{P_1} x_{P_1} + G_{Q_2} e_2 \\ A_{Q_1} x_{Q_1} + B_{Q_1} C_{Q_2} \tilde{x}_{Q_2} + G_{Q_1} e_1 \\ A_{Q_2} \tilde{x}_{Q_2} + B_{Q_2} C_{P_1} x_{P_1} + G_{Q_2} e_2 \end{bmatrix} \in S_1
$$

and thus (since $C_{Q_2}x_{Q_2} = C_{Q_2}\tilde{x}_{Q_2}$ and $C_{P_1}x_{P_1} = C_{Q_1}x_{Q_1}$)

$$\begin{bmatrix} \dot{x}_{P_1} \\ \dot{\tilde{x}}_{Q_2} \\ \dot{x}_{Q_1} \\ \dot{x}_{Q_2} \end{bmatrix} = \begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}C_{Q_2}\tilde{x}_{Q_2} + G_{P_1}e_1 \\ A_{Q_2}\tilde{x}_{Q_2} + B_{Q_2}C_{P_1}x_{P_1} + G_{Q_2}e_2 \\ A_{Q_1}x_{Q_1} + B_{Q_1}C_{Q_2}x_{Q_2} + G_{Q_1}e_1 \\ A_{Q_2}x_{Q_2} + B_{Q_2}C_{P_1}x_{P_1} + G_{Q_2}e_2 \end{bmatrix} \in S_1^{\mathrm{sym}} .$$

Moreover, since $(x_{P_1}, x_{Q_2}, x_{Q_1}, \tilde{x}_{Q_2}) \in S_1$ it holds that $C_{P_1}x_{P_1} = C_{Q_1}x_{Q_1}$ which, together with (13), lets S_1^{sym} fulfil condition (ii) in Definition 2. For the same reason, also condition (iii) is fulfilled, namely that $H_{P_1}x_{P_1} = H_{Q_1}x_{Q_1}$ and $H_{Q_2}x_{Q_2} = H_{Q_2}\tilde{x}_{Q_2}$.
Since S_1 is a full simulation relation, $\Pi_{X_{P_1}X_{Q_2}}S_1 = \Pi_{X_{P_1}X_{Q_2}}S_1^{\mathrm{sym}} = \mathcal{X}_{P_1} \times \mathcal{X}_{Q_2}$ and thus S_1^{sym} is a full simulation relation of $\Sigma_{P_1}\|\Sigma_{Q_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$. □

LEMMA 2. *Given full simulation relations $S_i, i = 1, 2$ of $\Sigma_{P_1}\|\Sigma_{q_2}$ and $\Sigma_{Q_1}\|\Sigma_{P_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$, respectively, and define the following linear subspaces*

$$\begin{aligned} \bar{S}_1 \;&:=\; \{(x_{P_1}, \bar{x}_{Q_2}, x_{Q_1}, -x_{Q_2}) \mid x_{Q_2}, \bar{x}_{Q_2} \in \ker C_{Q_2} \cap \\ &\quad \ker H_{Q_2}, x_{P_1} \in \ker C_{P_1} \cap \ker H_{P_1}, x_{Q_1} \in \\ &\quad \ker C_{Q_1} \cap \ker H_{Q_1}, (x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in S_1\} \\ \bar{S}_2 \;&:=\; \{(\bar{x}_{Q_1}, x_{Q_2}, -x_{Q_1}, x_{Q_2}) \mid x_{Q_1}, \bar{x}_{Q_1} \in \ker C_{Q_1} \cap \\ &\quad \ker H_{Q_1}, x_{P_2} \in \ker C_{P_2} \cap \ker H_{P_2}, x_{Q_2} \in \\ &\quad \ker C_{Q_2} \cap \ker H_{Q_2}, (x_{Q_1}, x_{Q_1}, \bar{x}_{Q_1}, x_{Q_2}) \in S_2\} \end{aligned}$$

Then $S_1 + \bar{S}_1$ and $S_2 + \bar{S}_2$ also define full simulation relations of $\Sigma_{P_1}\|\Sigma_{Q_2}$ and $\Sigma_{Q_1}\|\Sigma_{P_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$, respectively.

PROOF. Again, the statement will be proved only for $S_1 + \bar{S}_1$. Take any $(x_{P_1}, \bar{x}_{Q_2}, x_{Q_1}, -x_{Q_2}) \in \bar{S}_1$. Since all components fulfil $C_{P_1}x_{P_1} = C_{Q_1}x_{Q_1} = 0$, $C_{Q_2}x_{Q_2} = -C_{Q_2}\bar{x}_{Q_2} = 0$ and $H_{P_1}x_{P_1} = H_{Q_1}x_{Q_1} = 0$, $H_{Q_2}x_{Q_2} = -H_{Q_2}\bar{x}_{Q_2} = 0$, condition (ii) and (iii) in Proposition 1 is fulfilled. Since S_1 is a simulation relation, condition (i) in Proposition 1 ensures that there exists a $(w_{P_1}, w_{Q_2}, w_{Q_1}, \bar{w}_{Q_2}) \in S_1$ such that

$$\begin{bmatrix} A_{P_1}x_{P_1} \\ A_{Q_2}x_{Q_2} \\ A_{Q_1}x_{Q_1} \\ A_{Q_2}\bar{x}_{Q_2} \end{bmatrix} = \begin{bmatrix} w_{P_1} \\ w_{Q_2} \\ w_{Q_1} \\ \bar{w}_{Q_2} \end{bmatrix}$$

Note that since $(w_{P_1}, w_{Q_2}, w_{Q_1}, \bar{w}_{Q_2}) \in S_1$, $(w_{P_1}, \bar{w}_{Q_2}, w_{Q_1}, -w_{Q_2}) \in \bar{S}_1$. Hence

$$\begin{bmatrix} A_{P_1}x_{P_1} \\ A_{Q_2}\bar{x}_{Q_2} \\ A_{Q_1}x_{Q_1} \\ -A_{Q_2}x_{Q_2} \end{bmatrix} = \begin{bmatrix} w_{P_1} \\ \bar{w}_{Q_2} \\ w_{Q_1} \\ -w_{Q_2} \end{bmatrix}$$

which proves condition (i) in Proposition 1. Finally, fullness of S_1^{sym} follow from fullness of S_1. □

LEMMA 3. *Consider full simulation relation $(S_1 + \bar{S}_1)^{sym}$ and $(S_2 + \bar{S}_2)^{sym}$ of $\Sigma_{P_1}\|\Sigma_{Q_2}$ and $\Sigma_{Q_1}\|\Sigma_{P_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$ as defined in the previous lemmas. Then for every $x \in \ker C_{Q_2} \cap \ker H_{Q_2}$, $(0, x, 0, x) \in (S_1 + \bar{S}_1)^{sym}$ and analogously, for every $y \in \ker C_{Q_1} \cap \ker H_{Q_1}$, $(y, 0, y, 0) \in (S_2 + \bar{S}_2)^{sym}$.*

PROOF. Again, we will only prove the first half of the lemma. Since S_1 is a full simulation relation, it holds that for every $(0, x)$ there exists x_{Q_1}, x_{Q_2} such that $(0, x, x_{Q_1}, x_{Q_2}) \in S_1$ with $x_{Q_1} \in \ker C_{Q_1} \cap \ker H_{Q_1}$. If we take $x \in \ker C_{Q_2} \cap \ker H_{Q_2}$ then also $x_{Q_2} \in \ker C_{Q_2} \cap \ker H_{Q_2}$. Then $(0, x_{Q_2}, x_{Q_1}, -x) \in \bar{S}_1$ and therefore

$$\begin{bmatrix} 0 \\ x \\ x_{Q_1} \\ x_{Q_2} \end{bmatrix} - \begin{bmatrix} 0 \\ x_{Q_2} \\ x_{Q_1} \\ -x \end{bmatrix} = \begin{bmatrix} 0 \\ x - x_{Q_2} \\ 0 \\ x + x_{Q_2} \end{bmatrix} \in S_1 + \bar{S}_1$$

Moreover, $(0, x + x_{Q_2}, 0, x - x_{Q_2}) \in (S_1 + \bar{S}_1)^{\mathrm{sym}}$ and by the subspace property also

$$\begin{bmatrix} 0 \\ x - x_{Q_2} \\ 0 \\ x + x_{Q_2} \end{bmatrix} + \begin{bmatrix} 0 \\ x + x_{Q_2} \\ 0 \\ x - x_{Q_2} \end{bmatrix} = 2\begin{bmatrix} 0 \\ x \\ 0 \\ x \end{bmatrix} \in (S_1 + \bar{S}_1)^{\mathrm{sym}}$$

□

THEOREM 6. *For any given linear systems Σ_i, $i \in \{P_1, P_2, Q_1, Q_2\}$, circular assume-guarantee reasoning is sound, i.e.,*

$$\begin{array}{rccc} S_1: & \Sigma_{P_1} \parallel \Sigma_{Q_2} & \preccurlyeq & \Sigma_{Q_1} \parallel \Sigma_{Q_2} \\ S_2: & \Sigma_{Q_1} \parallel \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_1} \parallel \Sigma_{Q_2} \\ \hline S: & \Sigma_{P_1} \parallel \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_1} \parallel \Sigma_{Q_2} \end{array}$$

PROOF. Define a relation S in the following way:

$$\begin{aligned} S \;=\; &\{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) | \exists \tilde{x}_{Q_1}, \tilde{x}_{Q_2} : \\ &(x_{P_1}, x_{Q_2}, x_{Q_1}, \tilde{x}_{Q_2}) \in (S_1 + \bar{S}_1)^{\mathrm{sym}}, \\ &(x_{Q_1}, x_{P_2}, \tilde{x}_{Q_1}, x_{Q_2}) \in (S_2 + \bar{S}_2)^{\mathrm{sym}}\} \end{aligned} \tag{14}$$

with $(S_i + \bar{S}_i)^{\mathrm{sym}}, i = 1, 2$ constructed as in (11). Observe first that $(S_i + \bar{S}_i)^{\mathrm{sym}}$ are full simulation relations since S_i are full. Therefore, for every $(x_{P_1}, x_{Q_2}, x_{Q_1}, \tilde{x}_{Q_2}) \in (S_1 + \bar{S}_1)^{\mathrm{sym}}$ and every joint inputs $e_{P_1} = e_{Q_1} = e_1, e_{Q_2} = e_2$ it holds that

$$\begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}C_{Q_2}x_{Q_2} + G_{P_1}e_1 \\ A_{Q_2}x_{Q_2} + B_{Q_2}C_{P_1}x_{P_1} + G_{Q_2}e_2 \\ A_{Q_1}x_{Q_1} + B_{Q_1}C_{Q_2}\tilde{x}_{Q_2} + G_{Q_1}e_1 \\ A_{Q_2}\tilde{x}_{Q_2} + B_{Q_2}C_{Q_1}x_{Q_1} + G_{Q_2}e_2 \end{bmatrix} \in (S_1 + \bar{S}_1)^{\mathrm{sym}}$$

such that $H_{P_1}x_{P_1} = H_{Q_1}x_{Q_1}$ and $C_{P_1}x_{P_1} = C_{Q_1}x_{Q_1}$ as well as $H_{Q_2}x_{Q_2} = H_{Q_2}\tilde{x}_{Q_2}$ and $C_{Q_2}x_{Q_2} = C_{Q_2}\tilde{x}_{Q_2}$. For any $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$ there exists a $\tilde{x}_{Q_2}$ such that $(x_{P_1}, x_{Q_2}, x_{Q_1}, \tilde{x}_{Q_2}) \in (S_1 + \bar{S}_1)^{\mathrm{sym}}$. Taking

$$\dot{\tilde{x}}_{Q_2} = A_{Q_2}\tilde{x}_{Q_2} + B_{Q_2}C_{Q_1}x_{Q_1} + G_{Q_2}e_2$$

it is then straightforward to check that for any arbitrary joint inputs e_1, e_2

$$(\dot{x}_{P_1}, \dot{x}_{Q_2}, \dot{x}_{Q_1}, \dot{\tilde{x}}_{Q_2}) \in (S_1 + \bar{S}_1)^{\mathrm{sym}}$$

Similarly, by setting $\dot{\tilde{x}}_{Q_1} = A_{Q_1}\tilde{x}_{Q_1} + B_{Q_1}C_{Q_2}x_{Q_2} + G_{Q_1}e_1$ and observing that $C_{Q_1}x_{Q_1} = C_{Q_1}\tilde{x}_{Q_1}$ as well as $H_{Q_1}x_{Q_1} = H_{Q_1}\tilde{x}_{Q_1}$,

$$(\dot{x}_{Q_1}, \dot{x}_{P_2}, \dot{\tilde{x}}_{Q_1}, \dot{x}_{Q_2}) \in (S_2 + \bar{S}_2)^{\mathrm{sym}}$$

and therefore

$$(\dot{x}_{P_1}, \dot{x}_{P_2}, \dot{x}_{Q_1}, \dot{x}_{Q_2}) \in S .$$

Thus, S as defined in (14) is a simulation relation of $\Sigma_{P_1}\|\Sigma_{P_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$.
The next step is to prove that S is full. Since $(S_1 + \bar{S}_1)^{\mathrm{sym}}$ is a full simulation relation, there exists for every (x_{P_1}, x_{Q_2}) a $(\bar{x}_{Q_1}, \bar{x}_{Q_1})$ such that $(x_{P_1}, x_{Q_2}, \bar{x}_{Q_1}, \bar{x}_{Q_1}) \in (S_1 + \bar{S}_1)^{\mathrm{sym}}$. Moreover, since also $(S_2 + \bar{S}_2)^{\mathrm{sym}}$ is full, there exists for

an arbitrary x_{P_2} and the given $\bar{x}_{Q_1}$ a $(\hat{x}_{Q_1}, \hat{x}_{Q_2})$ such that $(\bar{x}_{Q_1}, x_{P_2}\hat{x}_{Q_1}, \hat{x}_{Q_2}) \in (S_2+\bar{S}_2)^{\text{sym}}$. Fullness of $(S_1+\bar{S}_1)^{\text{sym}}$ also ensures that there exists an element $(0, \hat{x}_{Q_2}, \tilde{x}_{Q_1}, \tilde{x}_{Q_2}) \in (S_1+\bar{S}_1)^{\text{sym}}$ with $\tilde{x}_{Q_1} \in \ker C_{Q_1}$. By Lemma 3, however, an element $(\tilde{x}_{Q_1}, 0, \tilde{x}_{Q_1}, 0)$ is contained in $(S_2+\bar{S}_2)^{\text{sym}}$. Hence

$$\begin{bmatrix} x_{P_1} \\ x_{Q_2} \\ \bar{x}_{Q_1} \\ \bar{x}_{Q_2} \end{bmatrix} + \begin{bmatrix} 0 \\ \hat{x}_{Q_2} - x_{Q_2} \\ \tilde{x}_{Q_1} \\ \tilde{x}_{Q_2} \end{bmatrix} = \begin{bmatrix} x_{P_1} \\ \hat{x}_{Q_2} \\ \bar{x}_{Q_1} + \tilde{x}_{Q_1} \\ \bar{x}_{Q_2} + \tilde{x}_{Q_2} \end{bmatrix} \in (S_1+\bar{S}_1)^{\text{sym}}$$

$$\begin{bmatrix} \bar{x}_{Q_1} \\ x_{P_2} \\ \hat{x}_{Q_1} \\ \hat{x}_{Q_2} \end{bmatrix} + \begin{bmatrix} \tilde{x}_{Q_1} \\ 0 \\ \tilde{x}_{Q_1} \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{x}_{Q_1} + \tilde{x}_{Q_1} \\ x_{P_2} \\ \hat{x}_{Q_1} + \tilde{x}_{Q_1} \\ \hat{x}_{Q_2} \end{bmatrix} \in (S_2+\bar{S}_2)^{\text{sym}}$$

from which the element

$$\begin{bmatrix} x_{P_1} \\ x_{P_2} \\ \hat{x}_{Q_1} + \tilde{x}_{Q_1} \\ \hat{x}_{Q_2} \end{bmatrix} \in S \tag{15}$$

can be constructed for any (x_{P_1}, x_{P_2}). $\square$

4. INTERCONNECTIONS WITH ALGEBRAIC CONSTRAINTS

In the first part, we were studying a feedback control like interconnection. This is appropriate in quite a few situations. Moreover, the interpretation of such an interconnection as a feedback control system is appealing, e.g. when applied to decentralized control problems. However, a different type of interconnection, resembling parallel composition as used for labeled transition systems and inducing algebraic constraints, also arises frequently in physical system interconnection.

Definition 3. *Given two linear dynamical systems Σ_i, $i = 1, 2$ of the form*

$$\Sigma_i: \quad \dot{x}_i = A_i x_i + B_i u_i \tag{16}$$

$$y_i = C_i x_i \tag{17}$$

where $x_i \in \mathcal{X}_i \subseteq \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^p$ and $y_i \in \mathbb{R}^q$.
Then the parallel composition $\Sigma_1 \|_{pc} \Sigma_2$ is given by

$$\Sigma_1 \|_{pc} \Sigma_2: \tag{18}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$y = C_1 x_1 = C_2 x_2$$

Equating the outputs of the parallel composition introduces the algebraic constraint $C_1 x_1 = C_2 x_2$, see Figure 5.

Thus, the equations (18) can be rewritten in so-called pencil form

$$\Sigma_{12}: \quad E_{12}\dot{z}_{12} = A_{12} z_{12}, z_{12} \in \mathcal{Z}_{12} \tag{19}$$

$$w_{12} = C_{12} z_{12}$$

where $\mathcal{Z}_{12} \subset \mathcal{X}_1 \times \mathcal{X}_2 \times \mathbb{R}_p$ and the matrices E_{12}, A_{12}, C_{12}

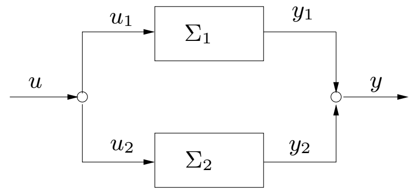

Figure 5: $\Sigma_1 \|_{\mathbf{pc}} \Sigma_2$

and the state and output vectors z_{12} and w_{12} are given by

$$z_{12} = \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix}, A_{12} = \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ C_1 & -C_2 & 0 \end{bmatrix}, \tag{20}$$

$$w_{12} = \begin{bmatrix} y \\ u \end{bmatrix}, E_{12} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, C_{12} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

respectively.
The formal definition and a linear algebraic characterization for simulation relations between DAE systems of the form (19) can be taken from [15].

Definition 4. *Consider a DAE system Σ_{12} of the form (19). Then the consistent subspace $\mathcal{V}_{12}^\star$ for Σ_{12} is the largest subspace $\mathcal{V}_{12} \subset \mathcal{Z}_{12}$ such that*

$$A_{12}\mathcal{V}_{12} \subset E_{12}\mathcal{V}_{12} \tag{21}$$

Definition 5. *Given two DAE systems Σ_i, $i = \{P_1P_2, Q_1Q_2\}$ of the form (19) with consistent subspaces $\mathcal{V}_i^\star$. Then a subspace $\tilde{S} \subset \mathcal{Z}_{P_1P_2} \times \mathcal{Z}_{Q_1Q_2}$ with $\Pi_{P_1P_2}\tilde{S} \subset \mathcal{V}_{P_1P_2}^\star$ is a simulation relation of $\Sigma_{P_1P_2}$ by $\Sigma_{Q_1Q_2}$ if and only if for all $(z_{P_1P_2}, z_{Q_1Q_2}) \in S$,*

1. *for all $\dot{z}_{P_1P_2} \in \mathcal{V}_{P_1P_2}^\star$ such that $E_{P_1P_2}\dot{z}_{P_1P_2} = A_{P_1P_2}z_{P_1P_2}$ there should exist a $\dot{z}_{Q_1Q_2} \in \mathcal{V}_{Q_1Q_2}^\star$ such that $E_{Q_1Q_2}\dot{z}_{Q_1Q_2} = A_{Q_1Q_2}z_{Q_1Q_2}$ and $(\dot{z}_{P_1P_2}, \dot{z}_{Q_1Q_2}) \in \tilde{S}$*
2. *$C_{P_1P_2}z_{P_1P_2} = C_{Q_1Q_2}z_{Q_1Q_2}$*

The simulation relation $\tilde{S}$ is full, *denoted by $\Sigma_{P_1P_2} \preccurlyeq \Sigma_{Q_1Q_2}$, if the projection on $\mathcal{Z}_{P_1P_2}$ is the consistent subspace, that is, $\Pi_{P_1P_2}\tilde{S} = \mathcal{V}_{12}^\star$.*

Theorem 7. *A subspace $\tilde{S} \subset \mathcal{Z}_{P_1P_2} \times \mathcal{Z}_{Q_1Q_2}$ is a simulation relation of $\tilde{\Sigma}_{P_1P_2}$ by $\tilde{\Sigma}_{Q_1Q_2}$ such that $\Pi_{P_1P_2}\tilde{S} \subset \mathcal{V}_{P_1P_2}^\star$ if and only if*

1. $\begin{bmatrix} \ker E_{P_1P_2} \cap \mathcal{V}_{P_1P_2}^\star \\ 0 \end{bmatrix} \subset \tilde{S} + \begin{bmatrix} 0 \\ \ker E_{Q_1Q_2} \cap \mathcal{V}_{Q_1Q_2}^\star \end{bmatrix}$
2. $\begin{bmatrix} A_{P_1P_2} & 0 \\ 0 & A_{Q_1Q_2} \end{bmatrix} \tilde{S} \subset \begin{bmatrix} E_{P_1P_2} & 0 \\ 0 & E_{Q_1Q_2} \end{bmatrix} \tilde{S}$
3. $\tilde{S} \subset \ker \begin{bmatrix} C_{P_1P_2} & -C_{Q_1Q_2} \end{bmatrix}$

Due to the special structure of the matrices A_i, E_i and $C_i, i \in \{P_1P_2, Q_1Q_2\}$ it is possible to reformulate Theorem 7 so that it is consistent with the definition of simulation relations for ODE systems as in Theorem 1.

Proposition 4. *There exists a simulation relation $S \subset \mathcal{X}_{P_1} \times \mathcal{X}_{P_2} \times \mathcal{X}_{Q_1} \times \mathcal{X}_{Q_2}$ of $\Sigma_{P_1}\|_{pc}\Sigma_{P_2}$ by $\Sigma_{Q_1}\|_{pc}\Sigma_{Q_2}$ if and only if for all $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$ and all $u \in \{v \mid \exists x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2} : (x_{P_1}, x_{P_2}, v)^T \in \mathcal{V}_{P_1P_2}^\star\}$ the following holds:*

1. $\begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}u \\ A_{P_2}x_{P_2} + B_{P_2}u \\ A_{Q_1}x_{Q_1} + B_{Q_1}u \\ A_{Q_2}x_{Q_2} + B_{Q_2}u \end{bmatrix} \in S$

2. $C_{P_1}x_{P_1} = C_{P_2}x_{P_2} = C_{Q_1}x_{Q_1} = C_{Q_2}x_{Q_2}$

PROOF. With the system matrices (20), condition 2. in Definition 5 yields

$$u_{P_1} = u_{Q_1} \tag{22}$$

and

$$C_{P_1}x_{P_1} = C_{Q_1}x_{Q_1} \tag{23}$$

Writing out 1. from Definition 5 results in

$$\begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}u_{P_1} \\ A_{P_2}x_{P_2} + B_{P_2}u_{P_1} \\ A_{Q_1}x_{Q_1} + B_{Q_1}u_{Q_1} \\ A_{Q_2}x_{Q_2} + B_{Q_2}u_{Q_1} \end{bmatrix} \in S \tag{24}$$

and

$$C_{P_1}x_{P_1} = C_{P_2}x_{P_2}, C_{Q_1}x_{Q_1} = C_{Q_2}x_{Q_2} \tag{25}$$

for all $(x_{P_1}, x_{P_2}, u_{P_1}, x_{Q_1}, x_{Q_2}, u_{Q_1}) \in \tilde{S}$ and

$$u_{P_1} \in \{v \mid \exists x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2} : (x_{P_1}, x_{P_2}, v)^T \in \mathcal{V}^\star_{P_1P_2}\} \tag{26}$$

Thus, equations (22) – (26) are equivalent to the conditions 1. and 2. in Proposition 4. □

To obtain linear algebraic conditions we first introduce the subspace $\mathcal{W}^\star_{12}$ as the projection of the consistent subspace $\mathcal{V}^\star_{12}$ on the state components x_1, x_2.

DEFINITION 6. *Let Σ_{12} be a DAE system of the form* (19) *and* (20)*. Then we denote by $\mathcal{W}^\star_{12}$ the subspace satisfying*

$$\mathcal{W}^\star_{12} = \Pi_{\mathcal{X}_1\mathcal{X}_2}\mathcal{V}^\star_{12} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \,\middle|\, \exists u : \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix} \in \mathcal{V}^\star_{12} \right\} \tag{27}$$

PROPOSITION 5. *There exists a simulation relation $S \subset \mathcal{X}_{P_1} \times \mathcal{X}_{P_2} \times \mathcal{X}_{Q_1} \times \mathcal{X}_{Q_2}$ of $\Sigma_{P_1}\|_{pc}\Sigma_{P_2}$ by $\Sigma_{Q_1}\|_{pc}\Sigma_{Q_2}$ if and only if the following conditions hold:*

1. $\mathrm{diag}\{A_{P_1}, A_{P_2}, A_{Q_1}, A_{Q_2}\}\, S \subset S$

2. $\mathrm{im} \begin{bmatrix} B_{P_1} \\ B_{P_2} \\ B_{Q_1} \\ B_{Q_2} \end{bmatrix} \cap \left(\mathcal{W}^\star_{P_1P_2} \times \mathcal{W}^\star_{Q_1Q_2}\right) \subset S$

3. $S \subset \ker \begin{bmatrix} C_{P_1} & -C_{P_2} & 0 & 0 \\ 0 & 0 & C_{Q_1} & -C_{Q_2} \\ C_{P_1} & 0 & -C_{Q_1} & 0 \end{bmatrix}$

PROOF. Condition 2. in Proposition 4 is equivalent to condition 3. in Proposition 5. Condition 1. in Proposition 4 results in

$$\mathrm{diag}\{A_{P_1}, A_{P_2}, A_{Q_1}, A_{Q_2}\}S + \mathrm{im} \begin{bmatrix} B_{P_1} \\ B_{P_2} \\ B_{Q_1} \\ B_{Q_2} \end{bmatrix} \subset S \tag{28}$$

but since u is restricted to

$$u \in \{v \mid \exists x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2} : (x_{P_1}, x_{P_2}, v)^T \in \mathcal{V}^\star_{P_1P_2}, \quad (x_{Q_1}, x_{Q_2}, v)^T \in \mathcal{V}^\star_{Q_1Q_2}\} \tag{29}$$

the image of the input map has to be restricted to the subspace of all admissible inputs. These are determined by the consistent subspaces to be

$$\{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid \exists u : (x_{P_1}, x_{P_2}, u) \in \mathcal{V}^\star_{P_1P_2}, (x_{Q_1}, x_{Q_2}, u) \in \mathcal{V}^\star_{Q_1Q_2}\} = \mathcal{W}^\star_{P_1P_2} \times \mathcal{W}^\star_{Q_1Q_2}$$

Therefore, conditions 2. and 3. in Proposition 5 are equivalent to condition 1. in Proposition 4. □

4.1 Compositional Reasoning

We begin our analysis for linear systems with algebraic constraints by examining the compositionality property for parallel composition.

THEOREM 8. *Given any four DAE systems Σ_i, $i \in \{P_1, P_2, Q_1, Q_2\}$ of the form (16) and* (20)*. Then parallel composition is compositional, i.e.*

$$\begin{array}{ccc} \Sigma_{P_1} \preccurlyeq \Sigma_{Q_1}, & & \Sigma_{P_2} \preccurlyeq \Sigma_{Q_2} \\ & \Longrightarrow & \\ \Sigma_{P_1}\|_{pc}\Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_1}\|_{pc}\Sigma_{Q_2} \end{array} \tag{30}$$

PROOF. Construct the relation S from given full simulation relations S_1 and S_2 of Σ_{P_1} and Σ_{P_2} by Σ_{Q_1} and Σ_{Q_2} as

$$S = \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid (x_{P_1}, x_{Q_1}) \in S_1, (x_{P_2}, x_{Q_2}) \in S_2\}$$

Then for any $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$ and any joint input $u \in \{v | \exists x_{P_1}, x_{P_2} : (x_{P_1}, x_{P_2}, v) \in \mathcal{V}^\star_{P_1P_2}$

$$\begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}u \\ A_{P_2}x_{P_2} + B_{P_2}u \\ A_{Q_1}x_{Q_1} + B_{Q_1}u \\ A_{Q_2}x_{Q_2} + B_{Q_2}u \end{bmatrix} \in S$$

since

$$\begin{bmatrix} A_{P_i}x_{P_i} + B_{P_i}u \\ A_{Q_i}x_{Q_i} + B_{P_i}u \end{bmatrix} \in S_i, i = 1, 2 \tag{31}$$

for all $u \in \mathcal{U}$. Moreover, since $y_{P_1} = y_{Q_1}$ due to S_1 and $y_{P_2} = y_{Q_2}$ due to S_2 and $y_{P_1} = P_2$ as well as $y_{Q_1} = y_{Q_2}$ enforced by parallel composition, condition (ii) in Proposition 4 is also fulfilled which proves that S is indeed a simulation relation of $\Sigma_{P_1}\|_{\mathrm{pc}}\Sigma_{P_2}$ by $\Sigma_{Q_1}\|_{\mathrm{pc}}\Sigma_{Q_2}$.
To show that S as defined in 31 is full, observe that (31) holds for all u. Since both S_1 and S_2 are full, we can find for every $u \in \{v | \exists x_{P_1}, x_{P_2} : (x_{P_1}, x_{P_2}, v) \in \mathcal{V}^\star_{P_1P_2}$ and every $(x_{P_1}, x_{P_2}) \in \mathcal{W}^\star_{P_1P_2}$ elements x_{Q_1}, x_{Q_2} such that $(x_{P_i}, x_{Q_i}) \in S_i, i = 1, 2$ and thus $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$. □

The converse is in general not true since the consistent subspace $\mathcal{V}^\star_{P_1P_2}$ restricts the choice of inputs u depending on the states x_{P_1}, x_{P_2}.

EXAMPLE 2. *Consider the systems*

$$\begin{array}{ll} \Sigma_{P_1} : \dot{x}_{P_1} = u_{P_1} & \Sigma_{Q_1} : \dot{x}_{Q_1} = -u_{Q_1} \\ \quad y_{P_1} = x_{P_1} & \quad y_{Q_1} = x_{Q_1} \end{array} \tag{32}$$

and

$$\begin{aligned} \Sigma_{P_2} : \dot{x}_{P_2} &= u_{P_2} \\ y_{P_2} &= 2x_{P_2} \end{aligned} \tag{33}$$

There exists a full simulation relation S of $\Sigma_{P_1}\|_{pc}\Sigma_{P_2}$ by $\Sigma_{Q_1}\|_{pc}\Sigma_{P_2}$, for example

$$S = \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{P_2} \mid x_{P_1} = x_{Q_1}, x_{P_1} = 2x_{P_2}\} \quad (34)$$

with the consistent subspaces given by

$$\mathcal{V}^{\star}_{P_1P_2} = \mathrm{im}\begin{bmatrix} x \\ 2x \\ 0 \end{bmatrix} = \mathcal{V}^{\star}_{Q_1P_2} \quad (35)$$

However, there does not exist any simulation relation of Σ_{P_1} by Σ_{Q_1} since

$$\mathrm{im}\begin{bmatrix} B_{P_1} \\ B_{Q_1} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \not\subseteq \ker\begin{bmatrix} C_{P_1} & -C_{Q_1} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (36)$$

4.2 Decomposition of the specification

In practical applications, the desired system behavior is often determined as a global specification. In order to apply modular techniques such as compositional and assume guarantee reasoning, a strategy to decompose the global specification as an interconnection of local specifications is helpful.

Proposition 6. *For any system Σ_P and parallel compositions, it holds that*

$$\Sigma_P \preccurlyeq \Sigma_P\|_{pc}\Sigma_P \quad (37)$$

Proof. Construct a simulation relation S by setting all state variables to be the same,

$$S = \{(x_1, (x_2, x_3)) \mid x_1 = x_2 = x_3 \in \Sigma_P\} \quad (38)$$

Then, S defines a full simulation relation of Σ_P by $\Sigma_P\|\Sigma_P$ since the evolution remains within the constrained subspace $Cx_1 = Cx_2 = Cx_3$ for all times. □

Proposition 7. *For any two systems Σ_P, Σ_Q, it holds that under parallel composition,*

$$\Sigma_P\|_{pc}\Sigma_Q \preccurlyeq \Sigma_P \quad (39)$$

Proof. The relation

$$S = \{((x_P, x_Q), \bar{x}_P) \mid x_P = \bar{x}_P, (x_P, x_Q) \in \mathcal{W}^{\star}_{PQ}\}$$

defines a full simulation relation of $\Sigma_P\|_{\mathrm{pc}}\Sigma_Q$ by Σ_P. □

The main result to decompose a given global specification Σ_Q into an interconnection of local specifications Σ_{Q_1} and Σ_{Q_2} can be stated as follows:

Theorem 9. *Given a specification Σ_Q and systems $\Sigma_{Q_i}, i = 1, 2$ of the form (16). Then decomposition of the specification*

$$\Sigma_Q \preccurlyeq \Sigma_{Q_1}\|_{pc}\Sigma_{Q_2} \quad (40)$$

is equivalent to

$$\Sigma_Q \preccurlyeq \Sigma_{Q_1}, \Sigma_Q \preccurlyeq \Sigma_{Q_2} \quad (41)$$

Proof. $\Longrightarrow$: Given a full simulation relation of Σ_Q by $\Sigma_{Q_1}\|_{\mathrm{pc}}\Sigma_{Q_2}$, Proposition 7 allows us to conclude that

$$\Sigma_Q \preccurlyeq \Sigma_{Q_1}\|_{\mathrm{pc}}\Sigma_{Q_2} \preccurlyeq \Sigma_{Q_1} \implies \Sigma_Q \preccurlyeq \Sigma_{Q_1}$$

and by symmetry,

$$\Sigma_Q \preccurlyeq \Sigma_{Q_1}\|_{\mathrm{pc}}\Sigma_{Q_2} \preccurlyeq \Sigma_{Q_2}\|_{\mathrm{pc}}\Sigma_{Q_1} \preccurlyeq \Sigma_{Q_2} \implies \Sigma_Q \preccurlyeq \Sigma_{Q_2}$$

$\Longleftarrow$: Compositionality and Proposition 6 yield

$$\Sigma_Q\|_{\mathrm{pc}}\Sigma_Q \preccurlyeq \Sigma_{Q_1}\|_{\mathrm{pc}}\Sigma_{Q_2}, \Sigma_Q \preccurlyeq \Sigma_Q\|_{\mathrm{pc}}\Sigma_Q \implies \Sigma_Q \preccurlyeq \Sigma_{Q_1}\|_{\mathrm{pc}}\Sigma_{Q_2}$$

□

Theorem 9 shows that when a given system $\Sigma_{P_1}\|_{\mathrm{pc}}\Sigma_{P_2}$ fulfils a global specification $\Sigma_Q = \Sigma_{Q_1}\|_{\mathrm{pc}}\Sigma_{Q_2}$ then it also fulfils parts of the specicfication, $\Sigma_{P_1}\|_{\mathrm{pc}}\Sigma_{P_2} \preccurlyeq \Sigma_{Q_i}, i = 1, 2$. Decomposition of the global specification Σ_Q into possibly smaller subsystems Σ_{Q_i} can thus simplify the overall verification task by applying compositional reasoning as in Theorem 8.

5. OUTLOOK

In this paper, we discussed compositional analysis techniques for linear dynamical systems. Adopting concepts from formal verification, it is possible to simplify verification tasks for control problems observing the modular structure of both the physical systems and the derived mathematical models. For the feedback interconnection of linear systems, we presented results for compositional analysis involving both non-circular and circular assume-guarantee reasoning rules. Complementary results for parallel compositions are obtained focussing mainly on decompositions of a given specification. Representing a specified property by a formal model that can then be related to the system model is common practice in computer science, for example in model checking ([2]). For linear continuous-time systems, a procedure of how to specify system properties such as stability or controllability as dynamical systems has not yet been developed. For a potential direction of research, consider the example of checking losslessness for a linear system of the form

$$\Sigma: \quad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad (42)$$

Σ is lossless if and only if there exists simulation relation between Σ and the one-dimensional non-linear system

$$\Xi: \quad \dot{\xi} = u^T y\ ,\ \xi \geq 0 \quad (43)$$

with external variables u and y. In fact, the map $\xi = \frac{1}{2}x^TQx$ represents a quadratic storage function for the system Σ. The simulation relation S of Σ by Ξ is then given by the graph

$$S = \{(x, \xi) \mid \xi = \frac{1}{2}x^TQx\} \quad (44)$$

Secondly, exploring the possibilities of decentralized control strategies seems fruitful within the presented framework.

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